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Induced representations of $U_q(so(3))$ with subrepresentations of integer spin only

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Abstract. We construct induced representations of $\mathcal{U} = U_q(so(3)) \cong U_q(sl(2))$ on suitable cosets of the matrix quantum group $SO_q(3)$. From these we obtain canonically finite-dimensional representations of $\mathcal{U}$ only of odd dimension, i.e. of integer spin. The matrix elements of these finite-dimensional representations are different from the standard $\mathcal{U}$ ones, which will be essential at least for the roots of unity case.

1. Introduction

From the papers of Drinfeld [1] and Jimbo [2] it is clear that the quantum algebras $U_q(sl(2))$ and $U_q(so(3))$ are isomorphic, since the constructions in [1, 2] use only information about the root systems of $sl(2) \equiv sl(2, \mathbb{C}) \cong so(3, \mathbb{C}) \equiv so(3)$.

On the other hand, the corresponding matrix quantum groups $SL_q(2)$ and $SO_q(3)$ are not isomorphic. More precisely, as in the classical case, the matrix quantum group $SL_q(2)$ is a double cover of $SO_q(3)$ [3]. Thus, one may expect that the induced holomorphic representations of $\mathcal{U} = U_q(sl(2))$ realized on suitable cosets of $SO_q(3)$ will have the feature of usual $SO(3, \mathbb{C})$ holomorphic irreps of being integer spin only.

This is exactly what we achieve in the present paper. For applications it is also important that the matrix elements of these finite-dimensional representations are different from the standard $\mathcal{U}$ ones, which will be essential at least for the roots of unity case.

The procedure used in this paper was proposed by the first named author in 1993 (unpublished) on the example of $SL_q(2)$. In the text we refer to the paper [4], where this procedure was applied to $GL_q(n)$ and $SL_q(n)$, and from where the mentioned unpublished results may be recovered for $n = 2$.

2. Matrix quantum group $SO_q(3)$ and the dual $U_q(so(3))$

The matrix quantum group $\mathcal{A} = SO_q(n)$ is the q-deformed analogue of the complex Lie group $SO(n, \mathbb{C})$ [5]. It is generated by n^2 elements which may be collected in an $n \times n$ matrix

$$T = (t_{ij}) \tag{2.1}$$

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and are subject to the following relations [5],

$$R_q T_1 T_2 = T_2 T_1 R_q \tag{2.2}$$

$$T C T^t C^{-1} = C T^t C^{-1} T = I_n \tag{2.3}$$

where R_q is a certain $n^2 \times n^2$ matrix, $T_1 = T \otimes I_n$, $T_2 = I_n \otimes T$, I_n is the identity $n \times n$ matrix, and C is a certain $n \times n$ matrix. The coalgebra structure is given by [5] the following formulae for the co-product $\delta_{\mathcal{A}}$, co-unit $\varepsilon_{\mathcal{A}}$, and antipode $\gamma_{\mathcal{A}}$,

$$\delta_{\mathcal{A}}(t_{ik}) = \sum_{j=1}^{n} t_{ij} \otimes t_{jk} \tag{2.4a}$$

$$\varepsilon_{\mathcal{A}}(t_{ik}) = \delta_{ik} \tag{2.4b}$$

$$\gamma_{\mathcal{A}}(T) = C\, T^t\, C^{-1} \tag{2.4c}$$

where the antipode is given in matrix form for compactness. Using this form, relations (2.3) are rewritten in the general form

$$T\, \gamma_{\mathcal{A}}(T) = \gamma_{\mathcal{A}}(T) T = I_n. \tag{2.5}$$

In the case $n = 3$ the R-matrix R_q has the form [5]

$$R_q = \left[\begin{array}{ccc|ccc|ccc} q &&&&&&&& \\ & 1 &&&&&&& \\ && q^{-1} &&&&&& \\ \hline & \lambda && 1 &&&&& \\ && \alpha && 1 &&&& \\ &&&&& 1 &&& \\ \hline && \beta && \alpha && q^{-1} && \\ &&&&& \lambda && 1 & \\ &&&&&&&& q \end{array}\right] \qquad \begin{array}{l} \lambda = q - q^{-1} \\ \alpha = -q^{-1/2}\lambda \\ \beta = (1 - q^{-1})\lambda \end{array} \tag{2.6}$$

and the matrix C is

$$C = \begin{pmatrix} 0 & 0 & q^{-1/2} \\ 0 & 1 & 0 \\ q^{1/2} & 0 & 0 \end{pmatrix} \qquad C^2 = I_3. \tag{2.7}$$

With these choices from (2.2) and (2.3) we can derive the explicit relations which the nine elements t_{ij} obey. We give them in an appendix since they will be necessary only in the next section.

The quantum algebra in duality with $SO_q(n)$ is $U_q(so(n))$. For $n = 3$ one has $\mathcal{U} = U_q(so(3)) \cong U_q(sl(2))$, cf [5]. We use a rational basis of $\mathcal{U}$ extracted from the L-operators of [5]. It differs from the basis of [2] by an algebraic transformation. In terms of this basis of $\mathcal{U}$, which we denote by $X^{\pm}$, $k^{\pm}$, the algebraic relations are

$$\begin{array}{ll} X^+X^- - X^-X^+ = (k^+ - k^-)/\lambda & \quad k^+k^- = k^-k^+ = 1_{\mathcal{U}} \\ k^{\pm}X^+ = q^{\mp 1}X^+k^{\pm} & \quad k^{\pm}X^- = q^{\pm 1}X^-k^{\pm} \end{array} \tag{2.8}$$

the coalgebra relations are

$$\begin{array}{l} \delta_{\mathcal{U}}(k^{\pm}) = k^{\pm} \otimes k^{\pm} \\ \delta_{\mathcal{U}}(X^+) = X^+ \otimes k^+ + 1_{\mathcal{U}} \otimes X^+ \\ \delta_{\mathcal{U}}(X^-) = k^- \otimes X^- + X^- \otimes 1_{\mathcal{U}} \end{array} \tag{2.9a}$$

$$\varepsilon_{\mathcal{U}}(k^{\pm}) = 1 \qquad \varepsilon_{\mathcal{U}}(X^+) = 0 \qquad \varepsilon_{\mathcal{U}}(X^-) = 0 \tag{2.9b}$$

$$\gamma_{\mathcal{U}}(k^{\pm}) = k^{\mp} \qquad \gamma_{\mathcal{U}}(X^+) = -X^+k^- \qquad \gamma_{\mathcal{U}}(X^-) = -k^+X^-. \tag{2.9c}$$

The duality between the algebras $\mathcal{U}$ and $\mathcal{A}$ is given by the pairings between the generators which follow from [5] (formula (2.1) for $k = 1$, up to renormalization); explicitly, we have

$$\langle X^+, T\rangle = \begin{pmatrix} 0 & -1 & 0 \\ 0 & 0 & q^{1/2} \\ 0 & 0 & 0 \end{pmatrix}$$
$$\langle X^-, T\rangle = \begin{pmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & -q^{-1/2} & 0 \end{pmatrix} \tag{2.10}$$
$$\langle k^\pm, T\rangle = \begin{pmatrix} q^\mp & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & q^\pm \end{pmatrix}.$$

These pairings are supplemented with the axiomatic pairing

$$\langle X, 1_\mathcal{A}\rangle = \varepsilon_\mathcal{U}(X) \qquad \forall X \in \mathcal{U}. \tag{2.11}$$

The pairing between arbitrary elements of $\mathcal{U}$ and $\mathcal{A}$ follows then from the properties of the duality pairing.

3. Representations of $U_q(so(3))$

Next we introduce the left regular representation of $\mathcal{U}$ which in the $q = 1$ case is the infinitesimal version of

$$\pi(M')M = M'^{-1}M \qquad M', M \in SO(3, \mathbb{C}) \tag{3.1}$$

namely we set

$$\pi_L(X)t_{ij} = \sum_{k=1}^{3} \langle \gamma_\mathcal{U}(X), t_{ik}\rangle t_{kj} \qquad X \in \mathcal{U}. \tag{3.2}$$

Note that in [4] was used the classical antipode $\gamma_\mathcal{U}^0$ (with deformation parameter set to classical values) instead of $\gamma_\mathcal{U}$, since in these cases things differ only in some intermediate formulae by inessential $q^{\cdots}$ factors. This would also be true here, but for uniformity we use $\gamma_\mathcal{U}$.

Explicitly, we obtain from (3.2) for the generators of $\mathcal{U}$

$$\pi_L(k^\pm)t_{ij} = q^{\pm(2-i)}t_{ij} \tag{3.3a}$$
$$\pi_L(X^+)T = \begin{pmatrix} t_{21} & t_{22} & t_{23} \\ -q^{-1/2}t_{31} & -q^{-1/2}t_{32} & -q^{-1/2}t_{33} \\ 0 & 0 & 0 \end{pmatrix} \tag{3.3b}$$
$$\pi_L(X^-)T = \begin{pmatrix} 0 & 0 & 0 \\ -t_{11} & -t_{12} & -t_{13} \\ q^{1/2}t_{21} & q^{1/2}t_{22} & q^{1/2}t_{23} \end{pmatrix}. \tag{3.3c}$$

In order to derive the action of π_L on arbitrary elements of the basis, we use the following twisted derivation rule consistent with the coproduct and the representation structure. Namely, we use [4]

$$\pi_L(y)ab = \hat{m}(\pi_L(\delta'_\mathcal{U}(y))(a \otimes b)) \tag{3.4}$$

where $\hat{m}$ is the multiplication map: $\hat{m}$: $\mathcal{A} \otimes \mathcal{A} \longrightarrow \mathcal{A}$, $\hat{m}(f \otimes f') = f \cdot f'$; $\delta'_\mathcal{U} = \sigma \circ \delta_\mathcal{U}$ is the opposite coproduct (σ is the permutation operator). Thus, in our concrete situation we have

$$\pi_L(k^\pm)ab = \pi_L(k^\pm)a \cdot \pi_L(k^\pm)b \tag{3.5a}$$

$$\pi_L(X^+)ab = \pi_L(k^+)a \cdot \pi_L(X^+)b + \pi_L(X^+)a \cdot b \tag{3.5b}$$

$$\pi_L(X^-)ab = \pi_L(X^-)a \cdot \pi_L(k^-)b + a \cdot \pi_L(X^-)b. \tag{3.5c}$$

Furthermore, we shall use the fact that π_L is a representation, i.e.

$$\begin{aligned} &\pi_L(ZZ') = \pi_L(Z) \cdot \pi_L(Z') \\ &\pi_L(\alpha Z + \beta Z') = \alpha\pi_L(Z) + \beta\pi_L(Z') \qquad \alpha, \beta \in \mathbb{C}. \end{aligned} \tag{3.6}$$

Next we introduce the right regular representation $\pi_R(X)$ [4] (which is also used in [6], although not given in this form, there being called left action and denoted by π_l):

$$\pi_R(X)t_{ij} = \sum_{k=1}^{3} t_{ik}\langle X, t_{kj}\rangle \qquad X \in \mathcal{U}. \tag{3.7}$$

Of course, as in [7] we shall use (3.7) as right action in order to reduce the left regular representation (and we could have also reversed the role of left and right).

Explicitly, we have

$$\pi_R(k^\pm)t_{ij} = q^{\pm(j-2)}t_{ij} \tag{3.8a}$$

$$\pi_R(X^+)T = \begin{pmatrix} 0 & -t_{11} & q^{1/2}t_{12} \\ 0 & -t_{21} & q^{1/2}t_{22} \\ 0 & -t_{31} & q^{1/2}t_{32} \end{pmatrix} \tag{3.8b}$$

$$\pi_R(X^-)T = \begin{pmatrix} t_{12} & -q^{-1/2}t_{13} & 0 \\ t_{22} & -q^{-1/2}t_{23} & 0 \\ t_{32} & -q^{-1/2}t_{33} & 0 \end{pmatrix}. \tag{3.8c}$$

The twisted derivation rule (cf [4, 6]) is now given by

$$\pi_R(y)ab = \hat{m}(\pi_R(\delta_{\mathcal{U}_g}(y))(a \otimes b)) \tag{3.9}$$

i.e. in our concrete situation

$$\pi_R(k^\pm)ab = \pi_R(k^\pm)a \cdot \pi_R(k^\pm)b \tag{3.10a}$$

$$\pi_R(X^+)ab = \pi_R(X^+)a \cdot \pi_R(k^+)b + a \cdot \pi_R(X^+)b \tag{3.10b}$$

$$\pi_R(X^-)ab = \pi_R(k^-)a \cdot \pi_R(X^-)b + \pi_R(X^-)a \cdot b. \tag{3.10c}$$

Furthermore, we note that since π_R is a representation we have

$$\begin{aligned} &\pi_R(ZZ') = \pi_R(Z) \cdot \pi_R(Z') \\ &\pi_R(\alpha Z + \beta Z') = \alpha\pi_R(Z) + \beta\pi_R(Z') \qquad \alpha, \beta \in \mathbb{C}. \end{aligned} \tag{3.11}$$

To continue further we need a PBW basis for $\mathcal{A}$. Due to the fact that there are many relations between the nine generators t_{ij}, there are several ways to introduce such a basis [3]. In particular, one may use the 2-to-1 covering of $SO_q(3)$ by the matrix quantum group $SL_q(2)$ [3]. However, there is a more economic and simpler way to introduce such a basis via the use of a Gauss decomposition. Moreover, the approach of [7] would require the use of a Gauss decomposition anyway. To obtain this decomposition we suppose now that there exists an element t_{33}^{-1}. Explicitly, we have

$$T = \begin{pmatrix} t_{11} & t_{12} & t_{13} \\ t_{21} & t_{22} & t_{23} \\ t_{31} & t_{32} & t_{33} \end{pmatrix} = \begin{pmatrix} 1 & -q^{1/2}\xi & -[2]^{-1}\xi^2 \\ 0 & 1 & \xi \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} t_{33}^{-1} & 0 & 0 \\ 0 & \eta & 0 \\ 0 & 0 & t_{33} \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ -q^{-1/2}\zeta & 1 & 0 \\ -[2]^{-1}\zeta^2 & \zeta & 1 \end{pmatrix}$$

$$= \begin{pmatrix} t_{33}^{-1} + \xi\eta\zeta + q^{-2}[2]^{-2}\xi^2\zeta^2 t_{33} & -q^{1/2}\xi\eta - q^{-1}[2]^{-1}\xi^2\zeta t_{33} & -[2]^{-1}\xi^2 t_{33} \\ -q^{-1/2}\eta\zeta - q^{-2}[2]^{-1}\xi\zeta^2 t_{33} & \eta + q^{-1}\xi\zeta t_{33} & \xi t_{33} \\ -q^{-2}[2]^{-1}\zeta^2 t_{33} & q^{-1}\zeta t_{33} & t_{33} \end{pmatrix} \tag{3.12}$$

where

$$\begin{aligned} &\xi = t_{23}t_{33}^{-1} \qquad \zeta = t_{33}^{-1}t_{32} \qquad t = t_{33} \\ &\eta = t_{33}^{-1}d_{11} \qquad d_{11} = t_{22}t_{33} - qt_{23}t_{32} \\ &[n] = [n]_q = (q^{n/2} - q^{-n/2})/\lambda' \qquad \lambda' = q^{1/2} - q^{-1/2} \end{aligned} \tag{3.13}$$

and the following formulae are used to check (3.12):

$$\begin{aligned} &t_{33}d_{11} = d_{11}t_{33} \qquad t_{33}^2 = d_{11}^2 \qquad \Rightarrow \qquad \eta^2 = 1_{\mathcal{A}} \\ &\xi^2 = -[2]t_{13}t_{33}^{-1} \qquad t_{13}d_{11} = q^2\, d_{11}t_{13} \qquad t_{23}d_{11} = q\, d_{11}t_{23} \\ &\zeta^2 = -[2]t_{33}^{-1}t_{31} \qquad t_{31}d_{11} = q^2\, d_{11}t_{31} \qquad t_{32}d_{11} = q\, d_{11}t_{32} \\ &t_{23}d_{11}t_{32} = q^{-3}\{t_{11}t_{33} - 1_{\mathcal{A}} - q^2 t_{13}t_{31}\}t_{33}^2 \\ &t_{23}d_{11}t_{33}^{-1} = q^{1/2}t_{13}t_{32} - q^{-1/2}t_{12}t_{33} \\ &t_{33}^{-1}d_{11}t_{32} = q^{-1/2}t_{23}t_{31} - q^{1/2}t_{33}t_{21}. \end{aligned} \tag{3.14}$$

The above relations in turn are verified by use of the explicit form of the algebraic relations of $SO_q(3)$ which are given in the appendix.

Thus, we see that the relevant variables are ξ, η, t, ζ and so a possible PBW basis is

$$f = f_{m\epsilon p\ell} = \xi^m \eta^\epsilon t^p \zeta^\ell \qquad m, \ell \in \mathbb{Z}_+, \epsilon = 0, 1, p \in \mathbb{Z}. \tag{3.15}$$

The commutation relations in this basis are

$$\begin{aligned} &t\xi = q^{-1}\xi t \qquad t\eta = \eta t \qquad t\zeta = q^{-1}\zeta t \\ &\eta\xi = \xi\eta \qquad \zeta\xi = \xi\zeta \qquad \zeta\eta = \eta\zeta. \end{aligned} \tag{3.16}$$

We see that this basis is very convenient since it is almost commutative.

Following the procedure of [4] our representation spaces will have elements which are formal power series in the basis (3.15) obeying right covariance conditions. By abuse of the notion we shall call these elements functions; explicitly, we write

$$\tilde{\varphi} = \sum_{\substack{m,\ell \in \mathbb{Z}_+ \\ \epsilon = 0,1,\, p \in \mathbb{Z}}} \mu_{m\epsilon p\ell}\, \xi^m \eta^\epsilon t^p \zeta^\ell. \tag{3.17}$$

The right covariance conditions [4] for the holomorphic representations are with respect to X^-, k^+:

$$\pi_R(X^-)\tilde{\varphi} = 0 \tag{3.18a}$$

$$\pi_R(k^+)\tilde{\varphi} = q^r\tilde{\varphi} \tag{3.18b}$$

where r is a parameter to be specified later. Note that from (3.18*b*) it follows that $\pi_R(k^-)\tilde{\varphi} = q^{-r}\tilde{\varphi}$. First, we calculate

$$\pi_R(X^-)\begin{pmatrix} \xi & \eta \\ \zeta & t \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ -q^{1/2} & 0 \end{pmatrix} \tag{3.19}$$

which means that in order to fulfill (3.18*a*) our functions should not depend on the variable ζ, i.e. the functions become

$$\tilde{\varphi} = \sum_{\substack{m \in \mathbb{Z}_+ \\ \epsilon = 0,1,\, p \in \mathbb{Z}}} \mu_{m\epsilon p}\, \xi^m \eta^\epsilon t^p. \tag{3.20}$$

Note that the algebra $\mathcal{Y}_q$ with PBW basis $\xi^m\eta^\epsilon t^p$ may be viewed as the q-deformation of (the local coordinates submanifold of) the coset $\mathcal{Y} = SO(3,\mathbb{C})/G^-$, where G^- is the subgroup of lower diagonal matrices with main diagonal entries equal to 1. Furthermore, note the decomposition $\mathcal{Y}_q = \mathcal{Y}_q^0 \oplus \mathcal{Y}_q^1$, where $\mathcal{Y}_q^0$, $\mathcal{Y}_q^1$ are isomorphic subalgebras with bases $\xi^m t^p$, $\xi^m \eta t^p$, respectively.

Next we obtain by direct calculation

$$\pi_R(k^+)\xi^m\eta^\epsilon t^p = q^p \xi^m \eta^\epsilon t^p. \tag{3.21}$$

From the latter and (3.18*b*) it follows that in (3.20) there is no summation in p since $p = r$; consequently, the parameter r should be integer and our functions become

$$\tilde{\varphi} = \sum_{\substack{m\in\mathbb{Z}_+\\ \epsilon=0,1}} \mu_{m\epsilon}\xi^m\eta^\epsilon t^r \qquad r\in\mathbb{Z}. \tag{3.22}$$

Now we suppose that q is not a root of unity. We calculate the transformation action:

$$\pi_L(k^\pm)\xi^m\eta^\epsilon t^r = q^{\pm(m-r)}\xi^m\eta^\epsilon t^r \tag{3.23a}$$

$$\pi_L(X^+)\xi^m\eta^\epsilon t^r = -q^{m/2-1}[m]\xi^{m-1}\eta^\epsilon t^r \tag{3.23b}$$

$$\pi_L(X^-)\xi^m\eta^\epsilon t^r = q^{(1-m)/2}\frac{[2r-m]}{[2]}\xi^{m+1}\eta^\epsilon t^r. \tag{3.23c}$$

It is easy to check that $\pi_L(k^\pm)$ and $\pi_L(X^\pm)$ satisfy (2.8). Note that these transformations do not change the parameters r and ϵ, i.e. we have obtained representations parametrized by $r\in\mathbb{Z}$, $\epsilon = 0, 1$. However, we see that the parameter ϵ is fictitious since the transformation rules do not depend on it. Furthermore, the variable η is passive also with respect to the right action: $\pi_R(X^\pm)\eta = 0$, $\pi_R(k^\pm)\eta = \eta$. Thus, for fixed ϵ the representation acts in the q-coset $\mathcal{Y}_q^\epsilon$, i.e. our functions become

$$\varphi = \varphi(\xi,\eta,t) = \sum_{m\in\mathbb{Z}_+}\mu_m\xi^m\eta^\epsilon t^r \qquad r\in\mathbb{Z}, \epsilon = 0, 1. \tag{3.24}$$

For simplicity, we shall further set $\epsilon = 0$ and denote our functions as $\varphi(\xi, t)$. We denote the representation action by π_r, which in terms of the functions $\varphi(\xi,t)$ may be written as

$$\pi_r(k^\pm)\varphi(\xi,t) = q^{\mp r}T^\xi_{q^\pm}\varphi(\xi,t) \tag{3.25a}$$

$$\pi_r(X^+)\varphi(\xi,t) = -q^{-1}T^\xi_{q^{1/2}}D^\xi_q\varphi(\xi,t) \tag{3.25b}$$

$$\pi_r(X^-)\varphi(\xi,t) = \frac{q^{1/2}\xi}{\lambda}T^\xi_{q^{-1/2}}(q^r T^\xi_{q^{-1/2}} - q^{-r}T^\xi_{q^{1/2}})\varphi(\xi,t) \tag{3.25c}$$

$$T^\xi_q f(\xi) = f(q\xi) \qquad D^\xi_q f(\xi) = \frac{\xi^{-1}}{\lambda'}(T^\xi_{q^{1/2}} - T^\xi_{q^{-1/2}})f(\xi). \tag{3.26}$$

We denote with $\mathcal{C}_r$ the representation space of functions $\varphi(\xi,t)$ with covariance properties (3.18) and transformation laws (3.23) (with $\epsilon = 0$) and (3.25). For generic $q\in\mathbb{C}$ and $r\in\mathbb{Z}_+$ the representation π_r is reducible. Indeed, for $r\in\mathbb{Z}_+$ the representation space $\mathcal{C}_r$ has an invariant subspace $\mathcal{E}_r$ of dimension $2r+1$ consisting of the vectors $\xi^m t^r$ for $m = 0, 1, \ldots, 2r$, ($\xi^0 t^0 \equiv 1_\mathcal{A}$). The latter statement is obvious, as from (3.23*c*) it follows that $\pi_L(X^-)\xi^{2r}t^r = 0$. Thus, $\xi^{2r}t^r$ ($\xi^0 t^0 = 1_\mathcal{A}$) is the lowest weight vector, while t^r is the highest weight vector: $\pi_L(X^+)t^r = 0$.

Thus, the set of finite-dimensional representations of $\mathcal{U}$ obtained as subrepresentations of the elementary representations realized on the coset $\mathcal{Y}_q^0$ (or $\mathcal{Y}_q^1$) of $SO_q(3)$ is parametrized by non-negative integers and for fixed $r\in\mathbb{Z}_+$ the corresponding finite-dimensional representation is of dimension $2r+1$, i.e. all dimensions are *odd*.

The latter result should be put in contrast with the fact that the set of finite-dimensional representations of $\mathcal{U}$ obtained as subrepresentations of the elementary representations realized on cosets of $SL_q(2)$ is parametrized by non-negative integers and for fixed $r \in \mathbb{Z}_+$ the corresponding finite-dimensional representation is of dimension $r+1$, i.e. all integer dimensions are possible.

Thus, we recover the classical result that the finite-dimensional irreps of $SO(3,\mathbb{C})$ are only of *integer* spin $j \in \mathbb{Z}_+$, ($j = r$), and hence of *odd* dimension $2j+1$, while the finite-dimensional irreps of $SL(2)$ (which is a double covering group of $SO(3,\mathbb{C})$) are of (half-)integer spin $j \in \mathbb{Z}_+/2$, ($j = r/2$), and hence of any integer dimension $2j+1$. (Of course, physicists consider finite-dimensional irreps of $SO(3,\mathbb{C})$ also of half-integer spin, calling them two-valued irreps; moreover, infinitesimally such considerations are also mathematically correct since $so(3) \cong sl(2)$.)

Otherwise, other results are in parallel with the $SL_q(2)$ case. In particular, the finite-dimensional invariant subspace $\mathcal{E}_r$ discussed above is the kernel of an operator $\mathcal{I}_r$ intertwining the representations π_r and $\pi_{r'}$, i.e.

$$\mathcal{I}_r \pi_r(Y) = \pi_{r'}(Y)\mathcal{I}_r \qquad Y \in \mathcal{U} \tag{3.27}$$

where r' is expected to be $-r-1$. According to the general prescription [7] this operator should be given by $(\pi_R(X^+))^s$ where the parameter s is expected to be $2r+1$ (=dim $\mathcal{E}_r$). This can be checked directly. Indeed, let $s \in \mathbb{N}$ and let us suppose that $\varphi' = (\pi_R(X^+))^s\varphi \in \mathcal{C}_{r'}$. The latter means first (by right covariance (3.18*a*)) that $\pi_R(X^-)\varphi' = 0$. We calculate

$$\begin{aligned}
\pi_R(X^-)\varphi' &= \pi_R(X^-)(\pi_R(X^+))^s\varphi \\
&= [\pi_R(X^-), (\pi_R(X^+))^s]\varphi \\
&= \pi_R([X^-, (X^+)^s])\varphi \\
&= \pi_R([X^-, (X^+)^s])\varphi \\
&= \pi_R([s](X^+)^{s-1}(q^{(s-1)/2}k^- - q^{-(s-1)/2}k^+)/\lambda)\varphi \\
&= \frac{[s]}{\lambda}\pi_R((X^+)^{s-1})\pi_R(q^{(s-1)/2}k^- - q^{-(s-1)/2}k^+)\varphi \\
&= \frac{[s]}{\lambda}\pi_R((X^+)^{s-1})(q^{(s-1)/2-r} - q^{r-(s-1)/2})\varphi \\
&= \frac{[s][s-1-2r]}{[2]}\pi_R((X^+)^{s-1})\varphi.
\end{aligned} \tag{3.28}$$

For q not a root of unity the last quantity may be zero only for $s = 2r+1$, as expected. Moreover, we use the other condition of right covariance (3.18*b*), $\pi_R(k^+)\varphi' = q^{r'}\varphi'$, i.e.

$$\begin{aligned}
\pi_R(k^+)\varphi' &= \pi_R(k^+)(\pi_R(X^+))^s\varphi \\
&= \pi_R(k^+(X^+)^s)\varphi \\
&= \pi_R(q^{-s}(X^+)^s k^+)\varphi \\
&= q^{-s}\pi_R((X^+)^s)\pi_R(k^+)\varphi \\
&= q^{r-s}\pi_R((X^+)^s)\varphi = q^{r-s}\varphi' \\
&\Longrightarrow r' = r - s = -r-1.
\end{aligned} \tag{3.29}$$

Thus, indeed the intertwining operator $\mathcal{I}_r$ is (up to a multiplicative non-zero constant)

$$\mathcal{I}_r = \pi_R(X^+)^{2r+1}. \tag{3.30}$$

Finally, as in [7] we introduce the restricted functions $\hat{\varphi}(\xi)$ by the formula

$$\hat{\varphi}(\xi) = (A\varphi)(\xi) \equiv \varphi(\xi, 1_{\mathcal{A}}) = \sum_{m \in \mathbb{Z}_+} \mu_m \xi^m. \tag{3.31}$$

Note that the algebra $\mathcal{Z}$ with PBW basis ξ^m may be viewed as (the local coordinates submanifold of) the coset $SO(3,\mathbb{C})/B^-$, where $B^- = HG^-$ is the subgroup of lower diagonal matrices, H being the subgroup of diagonal matrices.

We denote the representation space of $\hat{\varphi}(\xi)$ by $\hat{\mathcal{C}}_r$ and the representation acting in $\hat{\mathcal{C}}_r$ by $\hat{\pi}_r$. Thus, the operator A acts from $\mathcal{C}_r$ to $\hat{\mathcal{C}}_r$. The properties of $\hat{\mathcal{C}}_r$ follow from the intertwining requirement for A [7]:

$$\hat{\pi}_r A = A\pi_r. \tag{3.32}$$

In particular, the representation action $\hat{\pi}_r$ on the basis ξ^m is given by

$$\hat{\pi}_r(k^\pm)\xi^m = q^{\pm(m-r)}\xi^m \tag{3.33a}$$

$$\hat{\pi}_r(X^+)\xi^m = -q^{m/2-1}[m]\xi^{m-1} \tag{3.33b}$$

$$\hat{\pi}_r(X^-)\xi^m = q^{(1-m)/2}\frac{[2r-m]}{[2]}\xi^{m+1}. \tag{3.33c}$$

In terms of the functions $\hat{\varphi}$ the representation $\hat{\pi}_r$ acts as

$$\hat{\pi}_r(k^\pm)\hat{\varphi}(\xi) = q^{\mp r}T^\xi_{q^\pm}\hat{\varphi}(\xi) \tag{3.34a}$$

$$\hat{\pi}_r(X^+)\hat{\varphi}(\xi) = -q^{-1}T^\xi_{q^{1/2}}D^\xi_q\hat{\varphi}(\xi) \tag{3.34b}$$

$$\hat{\pi}_r(X^-)\hat{\varphi}(\xi) = \frac{q^{1/2}\xi}{\lambda}T^\xi_{q^{-1/2}}(q^rT^\xi_{q^{-1/2}} - q^{-r}T^\xi_{q^{1/2}})\hat{\varphi}(\xi). \tag{3.34c}$$

These functions have the property that we can extend (3.33) and (3.34) for arbitrary complex r. For generic $q, r \in \mathbb{C}$ the representations $\hat{\pi}_r$ are irreducible. For generic $q \in \mathbb{C}$ and $r \in \mathbb{Z}_+/2$ the representations $\hat{\pi}_r$ are reducible. In the latter case all properties parallel the infinitesimal version of the classical case, i.e. on the coset $\mathcal{Z}$ the restricted representations of the algebra $\mathcal{U}$ may have subrepresentations also of half-integer spin. Otherwise, the description is as for $\mathcal{C}_r$: the representation space $\hat{\mathcal{C}}_r$ has an invariant subspace $\hat{\mathcal{E}}_r$ of dimension $2r+1$ consisting of the vectors ξ^m for $m = 0, 1, \ldots, 2r$ ($\xi^0 \equiv 1_\mathcal{A}$), ξ^{2r} being the lowest weight vector and $1_\mathcal{A}$ being the highest weight vector.

4. Outlook

In the present paper we have shown that the induced holomorphic representations of $\mathcal{U} = U_q(sl(2))$ realized on the cosets $\mathcal{Y}_q^\epsilon$ of $SO_q(3)$ have finite-dimensional subrepresentations only of odd dimension. Thus, we have obtained finite-dimensional irreps of $U_q(so(3))$ of integer spin only and have, therefore, recovered the feature of usual $SO(3,\mathbb{C})$ holomorphic irreps being of integer spin only.

What is also important is that the matrix elements in (3.23) and (3.33) are different from those of the usual finite-dimensional irreps of $U_q(sl(2))$. One may argue that this amounts to a change of basis, and indeed introducing $v_m = (1/\sqrt{[m]![2r-m]!})\xi^m$, one may bring the transformation rules to the standard expressions. However, such a transformation would break down for q being an Nth root of unity so that $N \leqslant 2r$. Indeed, if $q = \mathrm{e}^{2\pi \mathrm{i}n/N}$, $N \in \mathbb{N}+2$, $n = 1, \ldots, N-1$, then $q^N = 1$ and $[N]_q = \sin(n\pi)/\sin(n\pi/N) = 0$, and the above transformation becomes undefined for certain m. Thus, for continuity we shall keep these matrix elements also for generic q. These different matrix elements would lead to different coefficients in the tensor product decompositions, different $3j$, $6j$ symbols, etc, which would be essential at least for the roots of unity case. This is postponed to a following paper, where the compact real forms $SO_q(3,\mathbb{R})$ and $U_q(so(3,\mathbb{R}))$ (for real q) will also be considered.

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Appendix. Explicit form of the algebra relations of $SO_q(\mathbf{3})$

Here we give the relations which the nine elements t_{ij} obey and which follow from (2.2) and (2.3) using (2.6) and (2.7). This explicit form is also necessary for the verification of the Gauss decomposition of section 3. The relations are

$$
\begin{aligned}
&t_{ik}t_{i\ell} = q^{\ell-k}t_{i\ell}t_{ik} \qquad i = 1, 3, k < \ell \\
&t_{kj}t_{\ell j} = q^{\ell-k}t_{\ell j}t_{kj} \qquad j = 1, 3, k < \ell \\
&t_{ij}t_{k\ell} = q^{k+\ell-i-j}t_{k\ell}t_{ij} \qquad i < k, j > \ell \\
&t_{k\ell}t_{k+1,\ell+1} = t_{k+1,\ell+1}t_{k\ell} + \lambda t_{k,\ell+1}t_{k+1,\ell} \qquad k, \ell = 1, 2 \\
&t_{k,1}t_{k+1,3} = q\, t_{k+1,3}t_{k,1} + \lambda t_{k+1,1}t_{k,3} \qquad k = 1, 2 \\
&t_{1,k}t_{3,k+1} = q\, t_{3,k+1}t_{1,k} + \lambda t_{1,k+1}t_{3,k} \qquad k = 1, 2 \\
&t_{12}t_{32} = q(t_{32}t_{12} + \alpha t_{13}t_{31}) \\
&t_{21}t_{23} = q(t_{23}t_{21} + \alpha t_{13}t_{31}) \\
&t_{11}t_{33} = q^2t_{33}t_{11} + q\lambda(t_{13}t_{31} - 1) \\
&t_{12}t_{22} = t_{22}t_{12} + \alpha t_{21}t_{13} \qquad t_{22}t_{23} = t_{23}t_{22} + \alpha t_{13}t_{32} \\
&t_{21}t_{22} = t_{22}t_{21} + \alpha t_{12}t_{31} \qquad t_{22}t_{32} = t_{32}t_{22} + \alpha t_{31}t_{23} \\
&t_{12}^2 = -q^{-1}[2]t_{11}t_{13} \qquad t_{23}^2 = -q^{-1}[2]t_{13}t_{33} \qquad t_{12}t_{32} = t_{21}t_{23} \\
&t_{21}^2 = -q^{-1}[2]t_{11}t_{31} \qquad t_{32}^2 = -q^{-1}[2]t_{31}t_{33} \qquad t_{32}t_{12} = t_{23}t_{21}.
\end{aligned}
\qquad \text{(A1)}
$$

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